Universality and Decidability of Number-Conserving Cellular Automata

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Abstract

Number-conserving cellular automata (NCCA) are particularly interesting, both because of their natural appearance as models of real systems, and because of the strong restrictions that number-conservation implies. Here we extend the definition of the property to include cellular automata with any set of states in $\mathbb{Z}$, and show that they can be always extended to "usual" NCCA with contiguous states. We show a way to simulate any one dimensional CA through a one dimensional NCCA, proving the existence of intrinsically universal NCCA. Finally, we give an algorithm to decide, given a CA, if its states can be labeled with integers to produce a NCCA, and to find this relabeling if the answer is positive.

Key words: Cellular automata, Number-conserving systems, Universality

1 Introduction

A cellular automaton (CA) is a discrete dynamical system, where the nodes ("cells") of some regular lattice (usually $\mathbb{Z}^d$) are mapped to a finite set of states; the state of a cell at a time $t+1$ is determined by a local function that takes at inputs the states of the cell and its neighbors at t. Cellular automata have been widely used as models of dynamical systems in which the behavior is determined by local interaction between spatially fixed elements. An interesting particular class of CA is the class of *number-conserving* CA (NCCA): roughly speaking, these are CA where the states are represented as numbers, and the sum of the states over all cells remains constant when the states are updated. This conservation may help to prove some properties of the dynamics, and can be usually related to the conservation of some quantity

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in the system that is being modeled. In CA traffic models ([13]), for instance, states are interpreted as the number of indestructible particles located in a cell. In fact, an interpretation in terms of particles can be given to any NCCA [10].

In [3], Boccara and Fukś give a necessary and sufficient condition for a one-dimensional CA of two states to be number-conserving, and study all the NCCA with neighborhoods $\{l, \ldots, r\}$, $l + r \leq 4$. In [4], they give a necessary and sufficient condition that holds for one-dimensional CA of any number of states, and use it to study all the three-state NCCA for $l+r \leq 2$. In [7] Durand *et al.* formalize three different definitions of number-conservation and show their equivalence. They write the generalization of Boccara's condition to two dimensions, and hint on the d-dimensional case; the decidability of number-conservation is thus proved (provided that the numeric value of the states is given). Durand *et al.* also give some examples of NCCA in several classes of one-dimensional CA, intersecting the classifications of Kůrka ([9]) and Braga *et al.* ([5,6]), and prove the emptiness of the remaining classes. In [11] Morita and Imai prove the universality of the class of number-conserving reversible partitioned CA; in [12] Morita *et al.* embed a simple general computer in a reversible, number-conserving two-dimensional partitioned CA. We must remark that, despite the title of [11], these articles do not settle the universality of NCCA; partitioned CA can be recoded as normal CA, but the recoding does not, in general, preserve number-conservation.

In section 2 we give the definition of NCCA and recall the necessary and sufficient conditions of [4] and [7]; we generalize the definitions and the conditions to allow any finite subset of $\mathbb{Z}$ as the set of states. In section 3, we show how any NCCA with such set of states can be extended to a NCCA with a set of states of the usual form $\{0, 1, \ldots, q\}$. In section 4 we show that any one dimensional CA may be simulated by a one dimensional CA. This implies that NCCA are capable of universal computation, and the particular form of the simulation implies the even stronger property of intrinsical universality. Finally, section 5 addresses the question of deciding, for any CA, whether its states can be relabeled with different values in $\mathbb{Z}$ to make the CA number-conserving, thus showing the decidability of the number-conservation property in a wide sense.

2 Definitions, previous results, and generalization

The definition and results for NCCA have so far assumed a set of states of the form $S = \{0, 1, \ldots, q\}$. This makes sense for some applications, but in the general case, there is no reason for restricting the class this way: we will define them for any finite $S \subset \mathbb{Z}$. As we will see, this doesn't change the previous results, and turns out to be useful in the later sections.

Cellular automata: A *cellular automaton* F in the d-dimensional space is formally described by a tuple $F = (d, S, N, f)$, where $d \in \mathbb{N}$ is the dimension, S is a finite set of *states*, $N \subset \mathbb{Z}$ is a finite *neighborhood*, and $f : S^N \to S$ is a *local transition rule*[1]. Since the neighborhood can always be enlarged by ignoring additional neighbors, we may assume it to be an hypercube, say,

$$N = \{(a_1, \ldots, a_d) \subset \mathbb{Z}^d : -l_i \le a_i \le r_i \quad \forall\, i\}$$

for some non-negative integers $l_1, r_1, \ldots, l_d, r_d$. A *configuration* is an element $c \in S^{\mathbb{Z}^d}$. The dynamics of the system is discrete in time, and at each time step, the current configuration $c^t \in S^{\mathbb{Z}^d}$ determines the next one, c^{t+1}, through

$$c^{t+1}_{(i_1,\ldots,i_d)} \;=\; f(c^t_{|(i_1,\ldots,i_d)+N})$$

In this way the local function induces a global one, which we will denote with the name of the automaton: $F : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$.

For the next definitions we follow Ollinger [14]; his formalization of intrinsic universality follows an idea presented in Albert and Čulik II [1] and in Bartlett and Garzón [2].

Sub-automaton: Let F and G be two cellular automata with sets of states S_F and S_G respectively; we say that F is a *sub-automaton* of G if there is an injective map $\phi : S_F \to S_G$ such that $\phi \circ F(c) = G \circ \phi(c)$ for all $c \in S_F^{\mathbb{Z}^d}$. In other words, F is just the restriction of G to some configurations (up to state relabeling).

Scaling: Here we consider the case with $d = 1$. Let σ be the *shift* function, i.e., the function $\sigma : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ such that $\sigma(c)_i = c_{i+1}$. For some strictly positive integer m, the packing map o^m is the function $o^m : S^{\mathbb{Z}} \to (S^m)^{\mathbb{Z}}$ such that $o^m(c)_i = (c_{mi}, \ldots, c_{mi+m-1})$. Notice that both functions are bijective. If F is a CA with states S, a $\langle m, n, k\rangle$-rescaling of F is a cellular automaton $F^{\langle m,n,k\rangle}$ with states S^m which verifies $F^{\langle m,n,k\rangle}(c) = \sigma^k \circ o^m \circ F^n \circ (o^m)^{-1}(c)$ for all $c \in (S^m)^{\mathbb{Z}}$.

Simulation and universality: We say that a CA F *simulates* a CA G if G is a sub-automaton of some rescaling of F. A CA is said to be *intrinsically universal* if it simulates any other CA. This definition, thanks to the restricted definition of simulation, is stronger than the usual definition of computational universality, which asks for the ability to simulate a universal Turing machine.

[1] We restrict the definitions to what we use; for general theory of CA, consult [8].

Period: A *period* $p \in \mathbb{N}^d$ for a configuration $c \in S^{\mathbb{Z}^d}$ is a vector such that $c_{i+p(c)} = c_i$, $\forall i \in \mathbb{Z}^d$. It can be easily checked that a period is preserved trough the iterations of a CA: for any $F = (d, S, N, f)$, a period p for c will be a period for $c' = F(c)$. The expression $0 \leq k \leq p$ will denote the set of vectors $\{0, \ldots, p_1\} \times \ldots \times \{0, \ldots, p_d\}$.

Number conservation: Let $C_P(d, S)$ be the set of all the configurations in $S^{\mathbb{Z}^d}$ that admit a period (the –spatially– *periodic configurations*); for each $c \in C_P(d, S)$ choose a period $p(c)$. We say that a CA $F = (d, S, N, f)$ is *number-conserving* iff

$$\sum_{0 \leq k \leq p(c)} c_k = \sum_{0 \leq k \leq p(c)} F(c)_k \qquad \forall c \in C_P \tag{1}$$

Durand *et al.* consider CAs with S of the form $\{0, \ldots, q\}$, for which they discuss three different definitions of number-conservation, and show that the three are equivalent. The first is the one we just gave; the second asks that the sum over all $\mathbb{Z}^d$ be conserved, for all finite configurations (configurations where $c_i = 0$ for all but a finite number of $i \in \mathbb{Z}^d$). The third definition asks, for all $c \in S^{\mathbb{Z}^d}$, that

$$\lim_{n \to \infty} \frac{\mu_n(c)}{\mu_n(F(c))} = 1 \qquad \text{where } \mu_n(c) = \sum_{i \in \{-n, \ldots, n\}^d} c_i$$

Necessary and sufficient conditions:

Two previous results that we will use are the necessary and sufficient conditions for a CA to be number-conserving in one dimension (proved in [4]) and in two dimensions (proved in [7]). Both assume a set of states of the form $S = \{0, \ldots, q\}$, and explicitly include 0 in the equation. In fact, the only property of 0 which is used is its *quiescence*: we say that a state s is quiescent iff $f(\{s\}^N) = s$. It follows directly from the definition of number-conservation that all the states of a number-conserving CA are quiescent.

In order to show that $0 \in S$ is not required in those results, we will deduce the result for $d = 1$ again, without that condition. Consider $F = (1, S, N, f)$, with $N = \{-l, \ldots, r\}$. Let n be $n = l + r + 1$, let $a \in S$ be any state, and take any $(x_1, \ldots, x_n) \in S^n$. Consider the configuration c consisting of infinite repetitions of

$$x_1, x_2, \ldots, x_n, \underbrace{a, \ldots, a}_{n-1}$$

If we apply equation (1) to this configuration (for period $2n-1$), we obtain

$$(n-1)\,a + \sum_{k=1}^{n} x_k \;=\; \sum_{k=1}^{n-1} f(\underbrace{a,\dots,a}_{n-k}, x_1,\dots,x_k) + \sum_{k=1}^{n} f(x_k,\dots,x_n,\underbrace{a,\dots,a}_{k-1})$$

Replacing $x_1 = a$ in this equation, we get

$$\begin{aligned} n\,a + \sum_{k=2}^{n} x_k \;&=\; \sum_{k=1}^{n-1} f(\underbrace{a,\dots,a}_{n-k+1}, x_2,\dots,x_k) + f(a,x_2,\dots,x_n) \\ &\quad + \sum_{k=2}^{n} f(x_k,\dots,x_n,\underbrace{a,\dots,a}_{k-1}) \\ &=\; \sum_{k=1}^{n-2} f(\underbrace{a,\dots,a}_{n-k}, x_2,\dots,x_{k+1}) + f(a,x_2,\dots,x_n) \\ &\quad + \sum_{k=2}^{n} f(x_k,\dots,x_n,\underbrace{a,\dots,a}_{k-1}) + a \end{aligned}$$

where we used the fact that $f(a,\dots,a) = a$. Taking the difference between the two last equations, we have

$$\begin{aligned} x_1 - a \;=\; &\sum_{k=1}^{n-2} f(\underbrace{a,\dots,a}_{n-k}, x_1,\dots,x_k) - f(\underbrace{a,\dots,a}_{n-k}, x_2,\dots,x_{k+1}) \\ &+ f(a,x_1,\dots,x_{n-1}) - f(a,x_2,\dots,x_n) + f(x_1,\dots,x_n) - a \end{aligned}$$

i.e.,

$$f(x_1,\dots,x_n) = x_1 + \sum_{k=1}^{n-1} f(\underbrace{a,\dots,a}_{n-k}, x_2,\dots,x_{k+1}) - f(\underbrace{a,\dots,a}_{n-k}, x_1,\dots,x_k) \qquad (2)$$

This is a necessary and sufficient condition. If F verifies equation (2) for all $(x_1,\dots,x_n) \in S^n$, then it verifies (1): the terms in the brackets will cancel when the sum ranges over the whole configuration, leaving just the sum of states before (on the left) and after (on the right) the application of F. Hence, the theorem reads:

Theorem 1 *Let $F = (1, S, N, f)$ be a CA with $S \subset \mathbb{Z}$, $N = \{-l,\dots,r\}$, $n = l + r + 1$. Let a be any state in S. Then, F is number-conserving if and only if f verifies equation (2) for all $(x_1,\dots,x_n) \in S^n$.*

The analogous condition for two dimensions was proved in [7] (for $a = 0$). The explicit writing of the proof is cumbersome, but the idea is the same as before: this time we consider a configuration with a matrix $(x_{i,j})_{i=1,\dots,m,j=1,\dots,n}$ (the size of the neighborhood) surrounded by $a's$, and write the necessary condition. Then, instead of subtracting the evaluation with $x_1 = a$ as before, we subtract the evaluation with $x_{1,\bullet} = a$ and the evaluation with $x_{\bullet,1} = a$, and add the evaluation with both $x_{1,\bullet} = a$ and $x_{\bullet,1} = a$. It is easy to see how the procedure can be further modified to get necessary and sufficient conditions for $d > 2$. For two dimensions, the resulting condition can be written as follows:

Theorem 2 *Let $F = (2, S, N, f)$ be a CA with $S \subset \mathbb{Z}$, $N = \{-l_1, \dots, r_1\} \times \{-l_2, \dots, r_2\}$, $n = l_1 + r_1 + 1$, $m = l_2 + r_2 + 1$. Let a be any state in S. Then F is number-conserving if and only if, for all $(x_{1,1}, \dots x_{m,n}) \in S^{nm}$, it satisfies*

$$\begin{aligned} f(M_{1,1,m,n}) \; &= \\ x_{1,1} \; &+ \sum_{i=1}^{m-1} f(M_{2,1,i+1,n}) - f(M_{1,1,i,n}) + \sum_{j=1}^{n-1} f(M_{1,2,m,j+1}) - f(M_{1,1,m,j}) \\ &+ \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f(M_{1,2,i,j+1}) + f(M_{2,1,i+1,j}) - f(M_{1,1,i,j}) - f(M_{2,2,i+1,j+1}) \end{aligned} \tag{3}$$

where $M_{T,L,B,R}$ represents a matrix filled with a's but for the bottom left corner, which is occupied with $(x_{i,j})_{i=T,\dots,B,j=L,\dots,R}$.

3 Extension from $S \subset \mathbb{Z}$ to $\{0, \dots, q\}$

The theorem of this section shows that the introduction of general $S \subset \mathbb{Z}$ as possible sets of states does not change the class of NCCA in any dramatic way: anything that can be seen in a NCCA with $S \in \mathbb{Z}$, can be seen in a NCCA with $\widetilde{S}$ of the form $\{0, \dots, q\}$, with the appropriate initial conditions. The theorem is stated and proved in one dimension, but we remark that versions of it for higher dimensions can be proved with similar arguments.

Theorem 3 *Let F be a NCCA with states $S \subset \mathbb{Z}$ and neighborhood of size n. Then F is a sub-automaton of a NCCA $\widetilde{F}$ with states $\widetilde{S} = \{0, \dots, \max S - \min S\}$ and neighborhood of size $2n$.*

Proof. Let F be a NCCA, $F = (1, S, N, f)$, with $N = \{-l, \dots, r\}$. By subtracting $\min S$ from all the states in S, we can assume that $S \subseteq \{0, \dots, M\}$, with $M = \max S - \min S$ and $0 \in S$, $M \in S$. In addition, without loss of generality, we may assume $l = 0$: if $l > 0$, we can apply the result to $F' = F \circ \sigma^{-l}$, obtaining $\widetilde{F'}$, and then take $\widetilde{F} = \widetilde{F'} \circ \sigma^l$.

We first note that for any $c \in S^{\mathbb{Z}}$,

$$\sum_{j=i-r}^{i} f(c_j, \ldots, c_{j+r}) \;\leq\; rM \;+\; c_i \tag{4}$$

To prove this assertion, notice that for this to be false, c_i^t must be strictly smaller than M; but in that case, we can change c, putting $c_i = M$, and we know, from the number-conservation, that the sum in $F(c)$ must increase accordingly. Since the only cells that can notice the change are those that "see" the state of i, we have that $\sum_{j=i-r}^{i} F(c)_j$ must increase in $M - c_i$. Since the new sum is bounded by $(r+1)M$, we get (4).

The procedure to get $\tilde{F}(c)$ from a configuration c is the following:

(1) $\forall i \in \mathbb{Z}\,,\; c'_i = \begin{cases} c_i & \text{for } c_i \in S \\ 0 & \sim \end{cases}$

(2) $c'' = F(c')$

(3) $\forall i \in \mathbb{Z}\,,\; c^0_i = \begin{cases} c''_i & \text{for } c_i \in S \\ c''_i + c_i & \sim \end{cases}$

(4) For $k = 1, \ldots, r$

$\forall i \in \mathbb{Z}\,,\; e^k_i = \max\{0, c^{k-1}_i - M\}$

$\forall i \in \mathbb{Z}\,,\; c^k_i = c^{k-1}_i - e^k_i + e^k_{i+1}$

(5) $\tilde{F}(c) = c^r$

So, we first remove c_i from each "invalid" position i, obtaining $c' \in S^{\mathbb{Z}}$. We apply F (which is a number-conserving operation), and then we put c_i back in its position, recovering the original sum of the states). At that point, some c^0_i may be greater than M. To correct this, the surplus at each site is pushed to the left, and this is done r times.

We claim that c^r verifies $0 \leq c^r_i \leq M$, for all i. Think of the c_i that are added in step 3 as particles labeled with i, and assume that each time an surplus is moved to the left (in step 4), preference is given to the particles with a higher label. Equation (4) assures that in the r sites to the left of i, there is enough place to accomodate the c_i particles:

$$(r+1)M - \sum_{j=i-r}^{i} c''_j \;=\; M + rM - \sum_{j=i-r}^{i} f(c'_j, \cdots, c'_{j+r}) \;\geq\; M \;\geq\; c_i$$

(remember that $c'_i = 0$ for the i we are considering). Notice that no particle is moving more than r steps. Therefore, the final state c^r_i is determined by c''_{i+k} and c_{i+k} for $k = 0, \ldots, r$; i.e., it is determined by c_{i+k} for $k = 0, \ldots, 2r$.

To define $\tilde{f}$, we just consider each $(x_0,\ldots,x_{2r}) \in \tilde{S}^{2r+1}$, apply the preceding procedure to the configuration c defined by $c_i = x_i$ for $i = 0,\ldots,2r$ and $c_i = 0$ elsewhere, and set $\tilde{f}(x_0,\ldots,x_{2r}) = c_0^r$. If $(x_0,\ldots,x_{2r}) \in S^{2r+1}$, then $c' = c$ in the procedure, there are no surplusses to move, and $\tilde{f}(x_0,\ldots,x_{2r}) = f(x_0,\ldots,x_r)$. $\square$

4 Universality

Theorem 4 *Let F be a CA with q states and neighborhood of size n. Then F can be simulated by a NCCA G with states $\{0,\ldots,2q+1\}$ and neighborhood of size $2n$.*

Proof. Let F be $F = (1, S, N, f)$, with $S = \{1,\ldots,q\}$ and $N = \{-l,\ldots,r\}$. We define $\hat{F} = (1, \hat{S}, \hat{N}, \hat{f})$ with

$$\hat{S} = \{-q,\ldots,-1,0,1,\ldots,q\} \qquad , \qquad \hat{N} = \{-2l-1,\ldots,2r+1\}$$

and $\hat{f}(a_{-2l-1},\ldots,a_{2r+1})$ given by

$$\begin{cases} f(a_{-2l}, a_{-2l+2}, \cdots, a_{2r}) & \text{if } a_{2i} = -a_{2i+1} > 0\,,\ -l \le i \le r \\ f(-a_{-2l}, -a_{-2l+2}, \cdots, -a_{2r}) & \text{if } a_{2i} = -a_{2i-1} < 0\,,\ -l \le i \le r \\ a_0 & \text{otherwise} \end{cases}$$

The idea is the following: each cell is split in two, and the new cells are occupied with a negative and a positive copy of the state the cell had. The iterations will follow the original CA rule, both on the negative and on the positive cells, and the sum will remain constant (zero).

With the definition given above, a positive cell will change its state only if it sees a "correct configuration" around it (a configuration of the form $a, -a, b, -b, \ldots$), and sees that its right neighbor is seeing it too; similarly, a negative cell will change only if it sees that its left –positive– neighbor will change in the same way (with the opposite sign). Hence, the only changes in the configuration are done in pairs of cells, and on each of this pairs the sum is conserved (and is 0). The CA $\hat{F}$ is then number-conserving. Through the injection $a \to (a, -a)$ F becomes a sub-automaton of $\hat{F}^{\langle 2,1,0\rangle}$. To avoid "negative" states, we add q to all the states of $\hat{F}$. $\square$

Corollary 5 *There are intrinsically universal one-dimensional NCCA.*

Proof. The relation of simulation is a preorder [14], and there exist intrinsically universal one-dimensional CA. $\square$

5 Decidability

Equations (2) and (3) show necessary and sufficient conditions for a CA in one and two dimensions, respectively, to be number-conserving; equations similar to them for higher dimensions are hard to write, but not to obtain. The decidabily of the property of number-conservation is thus proved, but with one important restriction: the numeric values of the states are taken as given. This is not a complete answer, since the labeling of the states in a CA is, in principle, arbitrary. If a model produces a CA with a set of states defined as colors, letters, or even numbers, *can we relabel the states with numbers such that the CA is number-conserving?* If the answer is yes, it would be interesting, since the conserved quantity could be traced back to the original system, or could help to prove some property of the CA. In particular, in [10] it is shown that NCCA can always be interpreted in terms of the interactions of indestructible particles; it would be interesting, for a system with states, say, *blue*, *yellow* and *red*, to know if the dynamics can be expressed in such terms.

If the set of states is required to be of the form $\{0, \ldots, q-1\}$, where $q = |S|$, then the answer to the problem is easy: just consider the $q!$ possible permutations of the labels, and check, for each of them, if the CA is number-conserving (using equations (2) or (3)). But this requirement is arbitrary: perhaps what we are seeing *can* be interpreted in terms of particles, but there is some quantity of particles which happens to never occur in a cell, and is therefore *absent* from our current set of states. In general, we would like to know if we can relabel the states of the CA with $|S|$ different elements of $\mathbb{Z}$, so as to make the CA number-conserving.

Example 1 Consider $F = (1, S, N, f)$ with $S = \{a, b, c\}$, $l = r = 3$, and f defined by:

$$f(x_0, x_1, x_2, x_3, x_4, x_5, x_6) = \begin{cases} c & \text{if} \quad (x_0 = x_1 = x_3 = x_4 = a \wedge x_2 = b) \\ & \vee \quad (x_1 = x_2 = x_4 = x_5 = a \wedge x_3 = b) \\ & \vee \quad (x_2 = x_3 = x_5 = x_6 = a \wedge x_4 = b) \\ x_3 & \sim \end{cases}$$

It can be checked, using equation (2), that all the bijections $\phi : \{a, b, c\} \to \{0, 1, 2\}$ produce rules that are not number-conserving. But F does become

number-conserving if we relabel its states with

$$\phi(a) = 0 \qquad \phi(b) = 3 \qquad \phi(c) = 1$$

It may be interpreted in terms of particles: if three particles are in a cell and the two neighboring cells in each direction are empty, then they separate; one goes to the left, one to the right, and one stays in the cell.

Theorem 6 *Let $F = (d, S, N, f)$ be a CA. Then there is an algorithm to decide if there exists a relabeling that makes F number-conserving, and to find it if the answer is positive.*

Proof. Write S as $S = \{s_0, s_1, \ldots, s_{|S|-1}\}$. We are looking for an injective function $\phi : S \to \mathbb{Z}$, such that F, redefined with states $\phi(S)$, becomes number-conserving. To find it solution, we will use the equation that gives the necessary and sufficient condition for the CA to be number-conserving; it will be equation (2), (3), or a form for higher dimensions, depending on the dimension in which F is defined. In [4], the equation was used to list the possible NCCAs, taking the values of the function f as the unknown variables. This time, the unknown variables of the equation are not the evaluations of the rule, but the numeric value of the states: the necessary and sufficient condition to make the relabeled version of f number-conserving is, in the one-dimensional case (the other cases are analogous), that

$$\begin{aligned} \phi(f(x_1, \ldots, x_n)) \;=\; \phi(x_1) + \sum_{k=1}^{n-1} \phi(f(\underbrace{s_0, \ldots, s_0}_{k}, x_2, \ldots, x_{n-k+1})) \\ -\phi(f(\underbrace{s_0, \ldots, s_0}_{k}, x_1, \ldots, x_{n-k})) \end{aligned} \tag{5}$$

for all $(x_1, x_2, \ldots, x_n) \in S^n$, where $n = |N|$. (Notice that from theorem 1, the choice of s_0 is arbitrary.) This is an homogeneous linear system of $|S|^n$ equations and $|S|$ variables. The set of solutions will be a linear subspace $V \in \mathbb{R}^{|S|}$. If $V = \{0\}$, the algorithm gives a negative answer. If $V \neq \{0\}$, we must still find out if it contains solutions *which have all the coordinates different* (if not, then there is no injection). In other words, we must check if $V \setminus \left(\bigcup_{j\neq k} E_{jk}\right) \neq \emptyset$, where E_{jk} is the hyperplane $x_j = x_k$. But

$$\begin{aligned} & V \setminus (\bigcup_{j\neq k} E_{jk}) = \emptyset \iff V = V \cap (\bigcup_{j\neq k} E_{jk}) = \bigcup_{j\neq k} V \cap E_{jk} \\ & \iff \exists j, k : V = V \cap E_{jk} \iff \exists j, k : V \subseteq E_{jk} \end{aligned}$$

where we use the fact that each $V \cap E_{jk}$ is a subspace of V, and a linear space

cannot be a finite union of proper subspaces [15]. The last condition can be easily checked by the algorithm, by adding the equation $\phi(x_j) = \phi(x_k)$ to the equation system and seeing if the space of solutions is the same.

If $V \setminus \left(\bigcup_{j \neq k} E_{jk}\right) \neq \emptyset$, then there is a solution ϕ to the problem, and it can be found in finite time. We can now relabel A with ϕ, making it number-conserving; if a set of states of the form $\{0, \ldots, q\}$ is desired, theorem 3 may be applied. $\square$

Note that the algorithm in theorem 6 is only needed when $|S| > 2$. For $|S| = 2$, if the CA can be made number-conserving, then it will be number-conserving for the two possible choices of $\phi : S \to \{0, 1\}$: if $S = \{a, b\} \subset \mathbb{Z}$ is a solution, then $\frac{S-a}{b-a} = \{0, 1\}$ is one too.

6 Conclusions

In this paper we develop several notions and results related to the class of number-conserving cellular automata, generalizing the definition and showing both the computational universality of the class and the decidability of the property that defines it (even for CA that are not initially expressed with numerical states).

The generalization of the definition allows the CA to have any set of states in $\mathbb{Z}$; we show that the necessary and sufficient conditions previously given for number-conservation do not require the previously assumed set of states of the form $\{0, 1, \ldots, q\}$. Furthermore, the generalization does not change the class in any profound way: in section 3 we show that for a NCCA with set of states $S \subset \mathbb{Z}$, the set of states can be completed to a contigous set of states, extending the rule while keeping the number-conservation property. Hence, any "generalized" NCCA can be seen as a "usual" NCCA, restricted to a subset of its configurations. The extension we show in theorem 3 requires an enlargement of the neighborhood; since we do not know examples where this enlargement is really needed, it may be the case that the result can be improved to an extension that preserves the neighborhood of the CA.

Section 4 shows that any one-dimensional CA can be simulated by a one-dimensional NCCA; the construction is very simple and proves that members of the latter of the latter class exhibit intrinsical universality, a property that implies computational universality.

Section 5 dealed with the question of deciding, for a given CA, whether its states can be labeled with integers, making it number-conserving. This ques-

tion turns out to be decidable, and we show how to find the relabeling, if there is one. The particular interest of this result is that it may help to reveal some conservative dynamics in an otherwise not-conserving CA; such a conservation may afterwards be interpreted in terms of indestructible particles [10].

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